

Equivalence of wave-particle duality to entropic uncertainty

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Interferometers capture a basic mystery of quantum mechanics: a single quantum particle can exhibit wave behavior, yet that wave behavior disappears when one tries to determine the particle's path inside the interferometer. This idea has been formulated quantitatively as an inequality, e.g., by Englert and Jaeger, Shimony, and Vaidman, which upper bounds the sum of the interference visibility and the path distinguishability. Such wave-particle duality relations (WPDRs) are often thought to be conceptually inequivalent to Heisenberg's uncertainty principle, although this has been debated. Here we show that WPDRs correspond precisely to a modern formulation of the uncertainty principle in terms of entropies, namely the min- and max-entropies. This observation unifies two fundamental concepts in quantum mechanics. Furthermore, it leads to a robust framework for deriving novel WPDRs by applying entropic uncertainty relations to interferometric models. As an illustration, we derive a novel relation that captures the coherence in a quantum beam splitter.

INTRODUCTION

When Feynman discussed the two-path interferometer in his famous lectures [1], he noted that quantum systems (quantons) display the behavior of both waves and particles and that there is a sort of competition between seeing the wave behavior versus the particle behavior. That is, when the observer tries harder to figure out which path of the interferometer the quanton takes, the wave-like interference becomes less visible. This tradeoff is commonly called wave-particle duality (WPD). Feynman further noted that this is "a phenomenon which is impossible ... to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery [of quantum mechanics]."

Many quantitative statements of this idea, so-called wave-particle duality relations (WPDRs), have been formulated [2–12]. Such relations typically consider the Mach-Zehnder interferometer for single photons, see Fig. 1. For example, a well-known formulation proven independently by Englert [2] and Jaeger et al. [3] quantifies the wave behavior by fringe visibility \mathcal{V} , and particle behavior by the distinguishability of the photon's path, \mathcal{D} . (See below for precise definitions; the idea is that "waves" have a definite phase, while "particles" have a definite location, hence \mathcal{V} and \mathcal{D} respectively quantify how definite the phase and location are inside the interferometer.) They found the tradeoff:

$$\mathcal{D}^2 + \mathcal{V}^2 \leq 1 \quad (1)$$

which implies $\mathcal{V} = 0$ when $\mathcal{D} = 1$ (full particle behavior means no wave behavior) and vice-versa, and also treats the intermediate case of partial distinguishability.

It has been debated, particularly around the mid-1990's [13–15], whether the WPD principle, also known as Bohr's complementarity principle, is equivalent to another fundamental quantum idea with no classical analog: Heisenberg's uncertainty principle [16]. The latter states that there are certain pairs of observables, such as

position and momentum or two orthogonal components of spin angular momentum, that cannot simultaneously be known or jointly measured. Likewise there are many quantitative statements of this idea, so-called uncertainty relations (URs) (see, e.g., [17–24]), and modern formulations typically use entropy instead of standard deviation as the uncertainty measure. See [19] for a survey of *entropic* uncertainty relations (EURs), and [25, 26] for reasons why entropy provides a more powerful framework for uncertainty.

At present the debate regarding wave-particle duality and uncertainty remains unresolved, to our knowledge. Yet Feynman's quote seems to suggest a belief that quantum mechanics has but one mystery and not two separate ones. In this article we lend quantitative support to this belief by showing a connection between URs and WPDRs, demonstrating that URs and WPDRs capture the same underlying physics [42]. This may come as a surprise, since Englert [2] originally argued that (1) "does not make use of Heisenberg's uncertainty relation in any form". To be fair, the uncertainty relation that we show is equivalent to (1) was not known at the time of Englert's paper, and was only recently discovered [20–24]. Specifically, we will consider EURs, where the particular entropies that are relevant to (1) are the so-called min- and max-entropies used in cryptography [27].

In what follows we consider several different WPDRs from the literature and show that they are in fact particular examples of EURs. Making this sort of connection not only unifies two fundamental concepts in quantum mechanics, but also means that novel WPDRs can be derived simply by applying already-proven EURs. As an illustration, we derive a novel WPDR for an exotic scenario involving a "quantum beam splitter" [28–30], where testing our WPDR would allow the experimenter to verify the beam splitter's quantum coherence.

Thus, in addition to unifying fundamental concepts, we provide a general framework for deriving and discussing WPDRs. We emphasize that the framework pro-

vided by EURs is highly robust, and entropies have well-characterized statistical meanings. Note that current approaches to deriving WPDRs often involve brute force calculation of the quantities one aims to bound; there is no general, elegant method currently in use. Our approach simply involves judicious application of the relevant uncertainty relation. What's more, we emphasize that uncertainty relations can be applied to interferometers in two different ways. One involves the principle of preparation uncertainty, which says that a quantum state cannot be prepared having low uncertainty for two complementary observables, and it turns out this principle is the one relevant to (1). The other involves the principle of measurement uncertainty, which says that two complementary observables cannot be jointly measured, and we discuss why this principle is actually what was tested in some recent interferometry experiments [29, 31]. Joint measurability in the context of interferometers was also discussed in [7, 32].

RESULTS

Our unified view associates a kind of behavior with the availability of a kind of information, or lack of behavior with missing information, as follows:

lack of particle behavior: $H_{\min}(Z|J)$

lack of wave behavior: $\min_{W \in XY} H_{\max}(W|K)$

where H_{\min} and H_{\max} are the min- and max-entropies (defined in Methods) which are commonly used in quantum information theory, Z is the path observable identified with the standard qubit basis (see Fig. 1), W is an orthonormal basis observable in the XY plane of the Bloch sphere [43], and J and K are some other quantum systems that help to reveal the behavior (e.g., J could be a which-path detector and K could be the quanton's internal degree of freedom). We formulate our general WPDR as

$$H_{\min}(Z|J) + \min_{W \in XY} H_{\max}(W|K) \geq 1 \quad (2)$$

which states that, for a two-path interferometer for single quantons, the sum of the ignorances about the particle and wave behaviors is lower bounded by 1 (i.e., 1 bit).

To be clear, (2) is explicitly an entropic uncertainty relation. The fact that it can be thought of as a WPDR, and furthermore that it encompasses the majority of WPDRs found in the literature for two-path single-quanton interferometers, is our result.

DISCUSSION

To illustrate this, we consider the celebrated Mach-Zehnder interferometer, shown schematically in Fig. 1. In

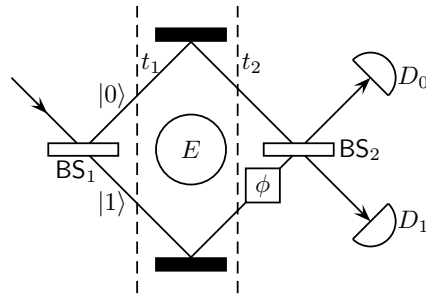


FIG. 1: Mach-Zehnder interferometer for single photons. Identifying $Z = \{|0\rangle, |1\rangle\}$ as the which-path basis, a superposition of these states is created at time t_1 . An environment E may then obtain some which-path information, e.g., E could be a gas of atoms whose internal states are affected by the presence of a photon. Finally at time t_2 a phase shift ϕ is applied to the lower arm and the two beams are recombined on a second beam splitter.

the simplest case one sends in a single photon towards a 50/50 (i.e., symmetric) beam splitter, BS_1 , which results in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, where $Z = \{|0\rangle, |1\rangle\}$ is the which-path basis, then a phase ϕ is applied to the lower arm giving the state $(|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$. Finally the two paths are recombined on a second 50/50 beam splitter BS_2 and the output modes are detected by detectors D_0 and D_1 . Visibility is then defined as

$$\mathcal{V} := \frac{p_{\max}^0 - p_{\min}^0}{p_{\max}^0 + p_{\min}^0} \quad (3)$$

where $p_{\max}^0 = \max_{\phi} \Pr(C = 0)$ and $p_{\min}^0 = \min_{\phi} \Pr(C = 0)$, where C denotes the random variable revealing which detector D_C clicks, with $C \in \{0, 1\}$. In this trivial example one has $\mathcal{V} = 1$. However many more complicated situations, for which the analysis is more interesting, have been considered in the extensive literature; we now illustrate how these situations fall under the umbrella of our framework with some illustrative examples.

Preparation uncertainty

\mathcal{P} - \mathcal{V} relation.—As a warm-up, we begin with the simplest known WPDR, the so-called predictability-visibility tradeoff. Predictability \mathcal{P} is defined as the *prior* knowledge, given the experimental setup, about which path the photon will take inside the interferometer. More precisely, $\mathcal{P} := 2p_{\text{guess}}(Z) - 1$ where $p_{\text{guess}}(Z)$ is the probability of correctly guessing Z . Non-trivial predictability is typically obtained by choosing BS_1 to be *asymmetric*. In such situations, the following bound holds [4, 5]:

$$\mathcal{P}^2 + \mathcal{V}^2 \leq 1. \quad (4)$$

This particularly simple example is a special case of Robertson's uncertainty relation involving standard deviations [32, 33]. However, Ref. [33] argues that (4) is

inequivalent to a family of EURs involving Rényi entropies, hence one gets the impression that entropic uncertainty is different from wave-particle duality, although Ref. [33] did not consider the EUR (2) involving the min- and max-entropies. For an arbitrary probability distribution $P = \{p_j\}$, the unconditional min- and max-entropies are given by $H_{\min}(P) = -\log \max_j p_j$ and $H_{\max}(P) = 2 \log \sum_j \sqrt{p_j}$ [44]. We find that Eq. (4) is equivalent to

$$H_{\min}(Z) + \min_{W \in XY} H_{\max}(W) \geq 1, \quad (5)$$

where the entropy terms are evaluated for the state at any time while the photon is inside the interferometer. It is straightforward to see that $H_{\min}(Z) = -\log \frac{1+\mathcal{D}}{2}$ and in the Methods we prove that

$$\min_{W \in XY} H_{\max}(W) = \log(1 + \sqrt{1 - \mathcal{V}^2}) \quad (6)$$

Plugging these relations into (5) gives (4).

\mathcal{D} - \mathcal{V} relation.—Let us move on to a more general and more interesting scenario where, in addition to prior which-path knowledge, one may obtain further which-path knowledge *during* the experiment due to the interaction of the photon with some environment E , which may act as a which-way detector. Most generally the interaction is given by a completely positive trace preserving (CPTP) map \mathcal{E} , with the input system being S at time t_1 and output systems being S and E at time t_2 , see Fig. 1. The final state is $\rho_{SE}^{(2)} = \mathcal{E}(\rho_S^{(1)})$, where we use the superscripts (1) and (2) to indicate the states at times t_1 and t_2 . We do not require \mathcal{E} to have any special form in order to derive our WPDR, so our treatment is slightly more general than [2], which derived (1) assuming the interaction is a path-preserving controlled unitary.

The path distinguishability is defined by $\mathcal{D} := 2p_{\text{guess}}(Z|E) - 1$, where $p_{\text{guess}}(Z|E)$ is the probability for correctly guessing the photon's path Z at time t_2 given that the experimenter performs the optimally helpful measurement on E . We find that (1) is equivalent to

$$H_{\min}(Z|E) + \min_{W \in XY} H_{\max}(W) \geq 1, \quad (7)$$

where the entropy terms are evaluated for the state $\rho_{SE}^{(2)}$. First, it is obvious from the operational meaning of the conditional min-entropy [27] that we have $H_{\min}(Z|E) = -\log p_{\text{guess}}(Z|E) = -\log \frac{1+\mathcal{D}}{2}$, and second we use our result (6) to rewrite (7) as (1). We remark that (1) and its entropic form (7) do not require BS_1 to be symmetric. This fact was emphasized in [2], hence we think of \mathcal{D} as accounting for both the prior Z knowledge associated with the asymmetry of BS_1 as well as the Z information gained from E . Also, the power of our approach should be clear from the fact that, unlike some previous approaches, we did not have to explicitly state the form of \mathcal{E} to derive the WPDR.

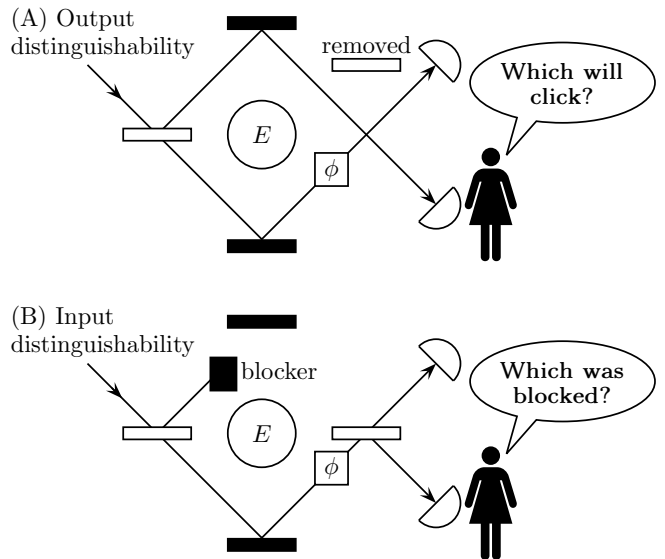


FIG. 2: Output versus input distinguishability. (A) Output distinguishability \mathcal{D} is measured by removing BS_2 and considering the probability of guessing correctly (given E) which detector will click. (B) Input distinguishability \mathcal{D}_i is measured by inserting a blocker into one of the interferometer arms and considering the probability of guessing correctly (given E and also C) which arm was blocked.

We now wish to make an important remark. The above analysis shows that Eqs. (1) and (4) correspond to applying the *preparation uncertainty relation* at time t_2 (just before the photon reaches BS_2). Preparation uncertainty restricts one's ability to predict the outcomes of *future* measurements of complementary observables. Thus, to experimentally measure the quantity \mathcal{P} or more generally \mathcal{D} , the experimenter *removes* BS_2 and sees how well he/she can guess which detector clicks, see Fig. 2A. (BS_2 can either be physically removed or effectively removed by exploiting another degree of freedom [28].) Of course, to then measure \mathcal{V} , the experimenter reinserts BS_2 in order to close the interferometer. (The fact that the apparatus must be modified to measure \mathcal{V} versus \mathcal{D} is one way of stating Bohr's complementarity principle.) Our point of emphasis is that this procedure falls into the general framework of preparation uncertainty.

Measurement uncertainty

On the other hand, uncertainty relations can be applied in a conceptually different way. Instead of two complementary output measurements and a fixed input state, consider a fixed output measurement and two complementary sets of input states. Namely consider the input ensembles $Z_i = \{|0\rangle, |1\rangle\}$ and $W_i = \{|w_+\rangle, |w_-\rangle\}$, where i stands for "input" and $|w_{\pm}\rangle = (|0\rangle \pm e^{i\phi}|1\rangle)/\sqrt{2}$ (see

Fig. 1 for definition of $|0\rangle$ and $|1\rangle$). The two Z_i inputs are generated by blocking the opposite arm of the interferometer, as in Fig. 2B, while the W_i states are generated by applying a phase (either 0 or π) to the lower arm. One can imagine this as a game, where Bob controls the input and Alice has control over both E and the detectors, Bob flips a coin to determine which path he will block in the case of Z_i (or which phase he will apply in the case of W_i) and Alice's goal is to guess the outcome of Bob's coin flip.

It may not be common knowledge that this scenario leads to a different class of WPDRs, therefore we illustrate the difference in Fig. 2. Furthermore, we refer to \mathcal{D} introduced above as output distinguishability, whereas in the present scenario we use the symbol \mathcal{D}_i and call this quantity input distinguishability, defined by

$$\mathcal{D}_i := 2p_{\text{guess}}(Z_i|EC) - 1,$$

where $p_{\text{guess}}(Z_i|EC) = 2^{-H_{\min}(Z_i|EC)}$ is Alice's probability to correctly guess Bob's Z_i state given that she has access to E and she knows which output detector clicks, stored in the random variable C [45]. Likewise we define the notion of *input* visibility \mathcal{V}_i via:

$$\mathcal{V}_i := [1 - (\min_{W \in XY} 2^{H_{\max}(W_i|C)} - 1)^2]^{1/2} \quad (8)$$

which quantifies how well Alice can determine W_i using C .

Now the uncertainty principle says that there is a tradeoff: if Alice can guess the Z_i states well then she cannot guess the W_i states well, and vice-versa. In other words, Alice's measurement apparatus, the apparatus to the right of the dashed line labeled t_1 in Fig. 1, cannot jointly measure Bob's Z and W observables. EURs involving von Neumann entropy have previously been applied to the joint measurement scenario [34, 35], we do the same for the min- and max-entropies to obtain

$$H_{\min}(Z_i|EC) + \min_{W \in XY} H_{\max}(W_i|C) \geq 1. \quad (9)$$

This can be rewritten as an explicit WPDR:

$$\mathcal{D}_i^2 + \mathcal{V}_i^2 \leq 1, \quad (10)$$

which can now be applied to a variety of situations.

Quantum BS₂.—As an interesting application of (9), we consider the scenario proposed in [28] and implemented in [29, 30], where the photon's polarisation P acts as a control system to determine whether or not BS₂ appears in the photon's path and hence whether the interferometer is open or closed, see Fig. 3. Since P can be prepared in an arbitrary input state $\rho_P^{(2)}$, such as a superposition, this effectively means that BS₂ is a "quantum beam splitter", i.e., it can be in a quantum superposition of being absent or present. The interaction coupling P to S is modelled as a controlled unitary as in Fig. 3. In

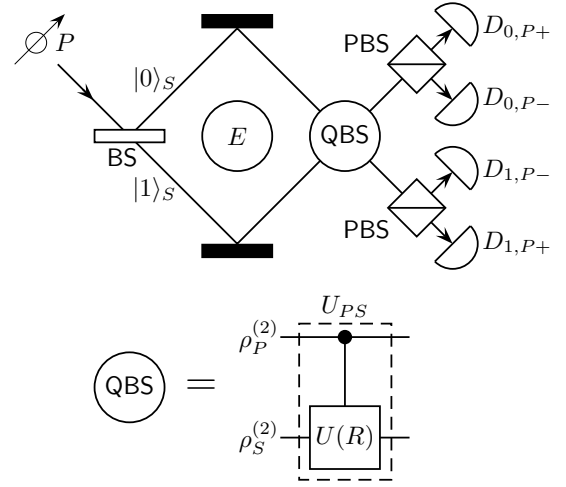


FIG. 3: In the quantum beam splitter (QBS) scenario, BS₂ can be in a superposition of "absent" and "present", as determined by the polarisation state $\rho_P^{(2)}$ at time t_2 . The QBS can be modelled as a controlled-unitary, $U_{PS} = |H\rangle\langle H|_P \otimes \mathbb{1}_S + |V\rangle\langle V|_P \otimes U(R)$, where $U(R)$ is the unitary on S associated with an asymmetric beam splitter with reflection probability R . Polarization-resolving detectors (PBS = polarizing beam splitter) on the output modes help to reveal the "quantumness" of the QBS.

this case we show (see Methods) that input and output visibility are equivalent:

$$\mathcal{V}_i = \mathcal{V} = 2|\kappa|\sqrt{R(1-R)}\langle V|\rho_P^{(2)}|V\rangle \quad (11)$$

where we assume the dynamics are path-preserving, i.e., $\mathcal{E}^S(|0\rangle\langle 0|) = |0\rangle\langle 0|$ and $\mathcal{E}^S(|1\rangle\langle 1|) = |1\rangle\langle 1|$, where $\mathcal{E}^S = \text{Tr}_E \circ \mathcal{E}$ is the reduced channel on S , and we denote the action on off-diagonal elements by $\mathcal{E}^S(|0\rangle\langle 1|) = \kappa|0\rangle\langle 1|$ where $|\kappa| \leq 1$. In (11), \mathcal{V} is evaluated for any pure state input $\rho_S^{(1)}$ from the XY plane of the Bloch sphere (e.g., $|+\rangle$). Now we apply the joint measurement relation (10) to this scenario and use (11) to obtain:

$$\mathcal{D}_i^2 + \mathcal{V}^2 \leq 1, \quad (12)$$

which extends a recent result in Ref. [12] to the case where E is non-trivial. This general treatment includes the special case where $\rho_P^{(2)} = |V\rangle\langle V|$, corresponding to a closed interferometer with an asymmetric BS₂. Ref. [31] experimentally tested this special case, under the assumption that E is trivial ($|\kappa| = 1$), in which case our visibility formula becomes $\mathcal{V}_i = \mathcal{V} = 2\sqrt{R(1-R)}$. We note that Ref. [31] did not remark that their experiment actually tested a relation different from (1), namely they tested a special case of (12).

Similarly, Ref. [29] tested (12) (again neglecting E) rather than (1), but they allowed $\rho_P^{(2)}$ to be in a superposition. At first sight this seems to test the WPDR in the case of a quantum beam splitter, but note that Eq. (9),

from which we derived (12), does not refer to the final polarisation P . The consequence is that neither the visibility \mathcal{V} nor the distinguishability \mathcal{D}_i depends on the phase coherence in $\rho_P^{(2)}$ and hence the data could be simulated by a classical mixture of BS₂ being absent or present [46]. Nevertheless, our framework provides a WPDR that captures the coherence in $\rho_P^{(2)}$ by conditioning either of the entropy terms in (9) on P after the photon exits the interferometer. For example, defining the polarisation-enhanced distinguishability: $\mathcal{D}_i^P := 2p_{\text{guess}}(Z_i|ECP) - 1$ gives the novel WPDR:

$$(\mathcal{D}_i^P)^2 + \mathcal{V}^2 \leq 1, \quad (13)$$

which captures the beam splitter's coherence (see Appendix) and could be tested with the setup in [29].

Conclusions

We have unified the wave-particle duality principle and the entropic uncertainty principle, showing that WPDRs are EURs in disguise. We leave it for future work to extend the connection of WPDRs to EURs for multiple photons [9] and multiple interference pathways [6]. We believe the framework presented here can be applied fairly universally to the case of single quantons in two-path interferometers. Indeed we show in a forthcoming paper [36] that the main results of both Ref. [8] (for asymmetric beam splitters) and Ref. [11] (for an internal degree of freedom) fall under our entropic uncertainty framework.

Our framework makes it clear how to formulate novel WPDRs by simply applying known EURs to novel interferometer models, and these new WPDRs will likely inspire new interferometry experiments. We also note that all of our relations hold if one replaces both min- and max-entropy with the well-known von Neumann entropy. Alternatively, one can use smooth entropies [24, 37], and the resulting smooth WPDRs may find application in the security analysis of interferometric quantum key distribution [38, 39].

METHODS

In what follows we first derive our preparation uncertainty WPDR, then our measurement uncertainty WPDR, and finally we discuss the QBS example. We begin by defining the relevant entropies. Let ρ_{AB} be any quantum state and $X = \{X_0, X_1\}$ be a binary POVM (positive operator valued measure) on A , then the classical-quantum min- and max-entropies are [47]:

$$H_{\min}(X|B) = 1 - \log(1 + \|\sigma_B^0 - \sigma_B^1\|_1), \quad (14)$$

$$H_{\max}(X|B) = \log(1 + 2\|\sqrt{\sigma_B^0}\sqrt{\sigma_B^1}\|_1), \quad (15)$$

where the 1-norm is $\|K\|_1 = \text{Tr}\sqrt{K^\dagger K}$, and $\sigma_B^j = \text{Tr}_A(X_j \rho_{AB})$ [43]. For any tripartite state $\rho_{AB_1B_2}$ where A is a qubit and letting Z and W be mutually unbiased bases on system A , we have [24]

$$H_{\min}(Z|B_1) + H_{\max}(W|B_2) \geq 1. \quad (16)$$

Preparation WPDR.—We apply Eq. (16) at time t_2 in Fig. 1 to obtain a preparation uncertainty WPDR:

$$H_{\min}(Z|E)_{\rho^{(2)}} + \min_{W \in XY} H_{\max}(W)_{\rho^{(2)}} \geq 1, \quad (17)$$

where the subscript $\rho^{(2)}$ indicates the entropy is evaluated for the state $\rho_{SE}^{(2)}$. Note that we place no assumptions on the dynamics prior to t_2 . We assume that after this time the photon reaches a symmetric (50/50) BS₂ and obtain the result:

$$\min_{W \in XY} H_{\max}(W)_{\rho^{(2)}} = \log(1 + \sqrt{1 - \mathcal{V}^2}). \quad (18)$$

To prove (18), we write $p_{\max}^0 = \max_{W \in XY} \text{Pr}(w_+)$, where we define $\text{Pr}(w_{\pm}) = \langle w_{\pm} | \rho_S^{(2)} | w_{\pm} \rangle$, and $|w_{\pm}\rangle$ are the two orthonormal basis states associated with the W observable. Suppose \tilde{W} is optimal such that $p_{\max}^0 = \text{Pr}(\tilde{w}_+)$, then due to the geometry of the Bloch sphere, we have $p_{\min}^0 = \text{Pr}(\tilde{w}_-)$. Thus, $p_{\max}^0 + p_{\min}^0 = 1$ and $p_{\max}^0 - p_{\min}^0 = \text{Tr}(\tilde{W} \rho_S^{(2)})$, which gives

$$\mathcal{V}^2 = [\text{Tr}(\tilde{W} \rho_S^{(2)})]^2 = \max_{W \in XY} \langle W \rangle_{\rho^{(2)}}^2.$$

Using $\text{Pr}(w_{\pm}) = (1 \pm \langle W \rangle)/2$ we write

$$\begin{aligned} H_{\max}(W)_{\rho^{(2)}} &= \log(1 + 2\sqrt{\text{Pr}(w_+) \text{Pr}(w_-)}) \\ &= \log(1 + \sqrt{1 - \langle W \rangle_{\rho^{(2)}}^2}). \end{aligned}$$

Thus we have

$$\min_{W \in XY} H_{\max}(W)_{\rho^{(2)}} = \log(1 + \sqrt{1 - \max_{W \in XY} \langle W \rangle_{\rho^{(2)}}^2})$$

giving the desired result (18). As noted in the main text, (18) allows us to rewrite (17) as $\mathcal{D}^2 + \mathcal{V}^2 \leq 1$.

Measurement WPDR.—To prove our joint measurement relation in (9), which considers a fixed output measurement and complementary input ensembles, we proceed as follows. The input ensembles $Z_i = \{|0\rangle_S, |1\rangle_S\}$ and $W_i = \{|w_+\rangle_S, |w_-\rangle_S\}$ can be generated by performing the relevant measurements on a reference system S' that is initially entangled to system S . Associating state ensembles with measurements on a reference system is a useful trick for deriving (9), as we shall see. Thus, we introduce a copy S' of S and consider a maximally entangled state $|\Phi\rangle_{S'S} = (|00\rangle + |11\rangle)/\sqrt{2}$. Now we feed S through the channel \mathcal{E} to obtain the state $\rho_{S'S'E} = (\mathcal{I} \otimes \mathcal{E})(|\Phi\rangle\langle\Phi|)$. Finally, to model the measurement on the output modes, we consider a channel

\mathcal{C} that measures the binary POVM $C = \{C_0, C_1\}$ on system S and stores an extra copy in C' , defined by $\mathcal{C}(\rho_S) = \sum_j \text{Tr}_S(C_j \rho_S) |j\rangle\langle j|_C \otimes |j\rangle\langle j|_{C'}$ [43]. The final state is then:

$$\tau_{S'CC'E} = (\mathcal{I} \otimes \mathcal{C} \otimes \mathcal{I})[(\mathcal{I} \otimes \mathcal{E})(|\Phi\rangle\langle\Phi|)], \quad (19)$$

which is often called the Choi-Jamiołkowski state. We apply (16) to this state to obtain

$$H_{\min}(Z_{S'}|EC)_\tau + \min_{W \in XY} H_{\max}(W_{S'}|C)_\tau \geq 1, \quad (20)$$

where $Z_{S'}$ and $W_{S'}$ are complementary observables on S' , and by symmetry we replaced C' by C . Since $|\Phi\rangle$ is maximally entangled, measuring $Z_{S'}$ corresponds to sending the states $\{|0\rangle_S, |1\rangle_S\}$ with equal probability through the interferometer, and similarly for $W_{S'}$ (with an inconsequential complication of taking the transpose of the W basis states). Realizing this, (20) becomes (9).

QBS example.—We now solve for the visibility in special case of a quantum BS₂. In this case the process of applying the unitary U_{PS} followed by measuring S can be viewed as measuring a binary POVM C on S , where the POVM elements are

$$C_0 = \text{Tr}_P[(\rho_P^{(2)} \otimes \mathbb{1})U_{PS}^\dagger(\mathbb{1} \otimes |0\rangle\langle 0|)U_{PS}], \quad (21)$$

$$C_1 = \text{Tr}_P[(\rho_P^{(2)} \otimes \mathbb{1})U_{PS}^\dagger(\mathbb{1} \otimes |1\rangle\langle 1|)U_{PS}]. \quad (22)$$

Specialising U_{PS} to be a controlled unitary implies that $\text{Tr}C_0 = \text{Tr}C_1 = 1$, and assuming the dynamics are path-preserving (see main text), we prove below that

$$\mathcal{V}_i = \mathcal{V} = |\kappa| r_\perp \quad (23)$$

where \mathcal{V} is evaluated for any pure state input $\rho_S^{(1)}$ from the XY plane of the Bloch sphere (e.g., $|+\rangle$), and $r_\perp := \max_{W \in XY} \text{Tr}(C_0 W)$ where the maximization is over all observables W in the XY plane. For example suppose we choose $U_{PS} = |H\rangle\langle H|_P \otimes \mathbb{1}_S + |V\rangle\langle V|_P \otimes U(R)$, where

$$U(R) = \begin{pmatrix} \sqrt{R} & \sqrt{1-R} \\ \sqrt{1-R} & -\sqrt{R} \end{pmatrix} \quad (24)$$

is the transformation for an asymmetric beam splitter. In this case we use Eq. (21) to obtain $r_\perp = 2\sqrt{R(1-R)}\langle V|\rho_P^{(2)}|V\rangle$, giving the result in (11).

To prove (23) we first rewrite p_{\max}^0 appearing in \mathcal{V} as

$$\begin{aligned} p_{\max}^0 &= \max_{W \in XY} \text{Pr}(C = 0|W_i = |w_+\rangle) \\ &= \text{Pr}(C = 0|\widetilde{W}_i = |\widetilde{w}_+\rangle), \end{aligned}$$

which supposes that $\widetilde{W}_i \in XY$ achieves the optimisation. Then we have $p_{\min}^0 = \text{Pr}(C = 0|\widetilde{W}_i = |\widetilde{w}_-\rangle)$. This is because we can think of $\text{Pr}(C = 0|W_i = |w_+\rangle)$ as the Hilbert-Schmidt inner product between C_0 and the density operator $\rho_S^{(2)}$, and the phase that minimizes this

inner product is 180 degrees added to the phases that maximizes it. Thus the output visibility is:

$$\begin{aligned} \mathcal{V} &= \frac{\text{Pr}(C = 0|\widetilde{W}_i = |\widetilde{w}_+\rangle) - \text{Pr}(C = 0|\widetilde{W}_i = |\widetilde{w}_-\rangle)}{\text{Pr}(C = 0|\widetilde{W}_i = |\widetilde{w}_+\rangle) + \text{Pr}(C = 0|\widetilde{W}_i = |\widetilde{w}_-\rangle)} \\ &= \text{Pr}(\widetilde{W}_i = |\widetilde{w}_+\rangle|C = 0) - \text{Pr}(\widetilde{W}_i = |\widetilde{w}_-\rangle|C = 0) \end{aligned} \quad (25)$$

where the second line used Bayes' rule:

$$\text{Pr}(C = 0|\widetilde{W}_i = |\widetilde{w}_+\rangle) = \frac{\text{Pr}(\widetilde{W}_i = |\widetilde{w}_+\rangle|C = 0) \text{Pr}(C = 0)}{\text{Pr}(\widetilde{W}_i = |\widetilde{w}_+\rangle)}$$

and assumed that $\text{Pr}(\widetilde{W}_i = |\widetilde{w}_+\rangle) = \text{Pr}(\widetilde{W}_i = |\widetilde{w}_-\rangle) = 1/2$. Now we have

$$\begin{aligned} &\sqrt{1 - \mathcal{V}^2} \\ &= \sqrt{4 \text{Pr}(\widetilde{W}_i = |\widetilde{w}_+\rangle|C = 0) \text{Pr}(\widetilde{W}_i = |\widetilde{w}_-\rangle|C = 0)} \\ &= \min_{W \in XY} \sqrt{4 \text{Pr}(W_i = |w_+\rangle|C = 0) \text{Pr}(W_i = |w_-\rangle|C = 0)} \\ &= \min_{W \in XY} 2^{H_{\max}(W_i|C=0)} - 1 \\ &= \min_{W \in XY} \frac{1}{2} 2^{H_{\max}(W_i|C=0)} + \frac{1}{2} 2^{H_{\max}(W_i|C=1)} - 1 \\ &= \min_{W \in XY} 2^{H_{\max}(W_i|C)} - 1 \\ &= \sqrt{1 - \mathcal{V}_i^2} \end{aligned} \quad (26)$$

which notes that $H_{\max}(W_i|C = 0) = H_{\max}(W_i|C = 1)$. Hence this proves $\mathcal{V} = \mathcal{V}_i$. Finally, we have

$$\mathcal{V} = \text{Tr}(\mathcal{E}^S(\widetilde{W})C_0) = \text{Tr}(|\kappa|\widetilde{W}C_0) = |\kappa| r_\perp. \quad (27)$$

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- [42] Some partial progress along this line was made in [32].
- [43] We use the same symbols (Z , W , X , C) for the observables as for the random variables they give rise to.
- [44] All logarithms are base 2.
- [45] Even though \mathcal{D} and \mathcal{D}_i are relevant to different physical scenarios, they become mathematically equivalent $\mathcal{D}_i = \mathcal{D}$ in the special case when C provides no which-path information and \mathcal{E} is path preserving.
- [46] Ref. [29] used other means to check for coherence.
- [47] It can be shown that these definitions are equivalent to those given in, e.g., [24, 27].

Appendix A: Testing coherence in a quantum beam splitter

Quantities sensitive to coherence

Here we further elaborate on the treatment of the quantum beam splitter (QBS) depicted in Fig. 3 of the main text. In particular we show that our novel WPDR stated in the main text:

$$(\mathcal{D}_i^P)^2 + \mathcal{V}^2 \leq 1 \quad (28)$$

captures the coherence of the beam splitter, whereas a weaker WPDR:

$$\mathcal{D}_i^2 + \mathcal{V}^2 \leq 1 \quad (29)$$

does not. Here the different distinguishabilities are

$$\mathcal{D}_i = 2p_{\text{guess}}(Z_i|C) - 1 \quad (30)$$

$$\mathcal{D}_i^P = 2p_{\text{guess}}(Z_i|CP) - 1 \quad (31)$$

where here P is the final polarisation after the QBS. For simplicity, we will neglect any interaction with an external environment E in what follows, and hence conditioning these distinguishabilities on E is not necessary. (Since we are interested in demonstrating that (28) can capture coherence, it suffices to demonstrate it for a special case where E plays no role.)

For comparison, we will also define “decohered” versions of \mathcal{D}_i and \mathcal{D}_i^P , $\mathcal{D}_i^{\text{dec}}$ and $\mathcal{D}_i^{P,\text{dec}}$ respectively, where the latter correspond to feeding in a decohered version of the polarisation state $\rho_P^{(2)}$, i.e., feeding in the corresponding classical mixture of $|H\rangle$ and $|V\rangle$ rather than a coherent superposition. Precisely this means replacing $\rho_P^{(2)}$ with $\mathcal{Z}(\rho_P^{(2)})$ where $\mathcal{Z}(\cdot) = |H\rangle\langle H|(\cdot)|H\rangle\langle H| + |V\rangle\langle V|(\cdot)|V\rangle\langle V|$ is the quantum channel that decoheres the polarisation state.

The first noteworthy point is that $\mathcal{D}_i = \mathcal{D}_i^{\text{dec}}$, hence measuring \mathcal{D}_i does not reveal the coherence in the QBS. This is because \mathcal{D}_i is not conditioned on P , so we evaluate it on the reduced state obtained from tracing over P , but tracing over P removes any dependence on the off-diagonal elements of $\rho_P^{(2)}$ in $\{|H\rangle, |V\rangle\}$ basis, since the unitary U_{PS} is controlled by the $\{|H\rangle, |V\rangle\}$ basis. Recall that the form of U_{PS} is

$$U_{PS} = |H\rangle\langle H|_P \otimes \mathbb{1}_S + |V\rangle\langle V|_P \otimes U(R). \quad (32)$$

where $U(R)$ was defined in (24).

On the other hand we show that, in general, $\mathcal{D}_i^P \neq \mathcal{D}_i^{P,\text{dec}}$, so \mathcal{D}_i^P has the potential to reveal coherence. We also remark that the following hierarchy holds in general:

$$\mathcal{D}_i \leq \mathcal{D}_i^{P,\text{dec}} \leq \mathcal{D}_i^P$$

The first inequality holds because $\mathcal{D}_i = \mathcal{D}_i^{\text{dec}}$ and conditioning on P can never decrease the guessing probability. The second inequality holds because the decoherence operation $\mathcal{Z}(\cdot)$ commutes with U_{PS} and hence can be viewed as restricting the class of measurements over which one optimises to evaluate the guessing probability.

For simplicity, let us consider a one-parameter family of input states $\rho_P^{(2)} = |\psi_P^{(2)}\rangle\langle\psi_P^{(2)}|$ with $|\psi_P^{(2)}\rangle = \cos\alpha|H\rangle + \sin\alpha|V\rangle$. Thus, we have an open interferometer when $\alpha = 0$ and a closed one when $\alpha = 90$ deg. Note that in this case the visibility becomes

$$\mathcal{V} = 2\sqrt{R(1-R)}\sin^2(\alpha).$$

Solving analytically for the different distinguishabilities (see below for derivation) gives:

$$\mathcal{D}_i = |1 - 2(\sin^2\alpha)(1-R)|, \quad (33)$$

$$\mathcal{D}_i^{P,\text{dec}} = 1 - (\sin^2\alpha) \cdot (1 - |2R - 1|), \quad (34)$$

$$\mathcal{D}_i^P = \sqrt{1 - 4R(1-R)(\sin^4\alpha)}. \quad (35)$$

Clearly these formulas indicate that, in general, $\mathcal{D}_i^P \neq \mathcal{D}_i^{P,\text{dec}}$, hence showing that \mathcal{D}_i^P reveals coherence. We can see a clear distinction between these distinguishabilities in Fig. 4, which considers the case of $R = 0.4$.

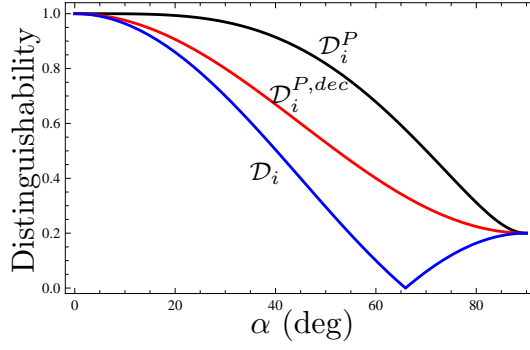


FIG. 4: Plot of \mathcal{D}_i , $\mathcal{D}_i^{P,dec}$, and \mathcal{D}_i^P as a function of α , for $R = 0.4$.

Discussion of Ref. [29]

We remark that \mathcal{D}_i^P could be measured using polarisation-resolving detectors on the output modes, as in Fig. 3 of the main text, assuming one chooses the optimal polarisation basis to measure on each output mode. (See below where we explicitly solve for the optimal polarisation basis to measure.) We note that the setup in Ref. [29] has polarisation-resolving detectors, and hence could measure \mathcal{D}_i^P . However, the procedure outlined in [29] for measuring distinguishability corresponds to measuring our \mathcal{D}_i . Figure 5 shows our theoretical predictions for the situation in [29], corresponding to $R = 0.5$. At first sight our predictions appear to disagree with [29] in the sense that our Fig. 5B, which plots \mathcal{V}^2 , \mathcal{D}_i^2 , and $\mathcal{V}^2 + \mathcal{D}_i^2$, looks very different from the corresponding plot of these quantities in Fig. 4 of [29]. An explanation for the disagreement is that [29] may have actually plotted \mathcal{V} , \mathcal{D}_i , and $\mathcal{V} + \mathcal{D}_i$ in their Fig. 4. Indeed, their Fig. 4 looks similar to our predictions for \mathcal{V} and \mathcal{D}_i in Fig. 5A, and we have $\mathcal{V} + \mathcal{D}_i = 1$ which is consistent with their Fig. 4. The authors of [29] have confirmed that their Fig. 4 plotted visibility and distinguishability as opposed to their squares [40]. We emphasise that this minor issue with their plot does not affect the conclusions of Ref. [29].

Since we predict that $\mathcal{V}^2 + \mathcal{D}_i^2$ can be strictly less than 1, then testing our novel relation $\mathcal{V}^2 + (\mathcal{D}_i^P)^2 \leq 1$ can give a more stringent test of wave-particle duality. Indeed we show that for the setup in [29], this relation is as strong as possible, i.e., it is satisfied with equality $\mathcal{V}^2 + (\mathcal{D}_i^P)^2 = 1$. This is depicted in Fig. 5C.

Derivation of distinguishability formulas

The derivation of Eqs. (33)-(35) proceeds as follows. We start with a maximally entangled state on SS' : $|\Phi\rangle_{SS'} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and the polarisation register in the state $|\psi\rangle_P = c|H\rangle + s|V\rangle$, where $c = \cos \alpha$ and $s = \sin \alpha$. Recall that U_{PS} is given by (32), and its action results in the state:

$$(U_{PS} \otimes \mathbb{1}_{S'})|\psi\rangle_P \otimes |\Phi\rangle_{SS'} = c|H\rangle_P \otimes \frac{|0\rangle_S|0\rangle_{S'} + |1\rangle_S|1\rangle_{S'}}{\sqrt{2}} + s|V\rangle_P \otimes \frac{U(R)|0\rangle_S|0\rangle_{S'} + U(R)|1\rangle_S|1\rangle_{S'}}{\sqrt{2}}.$$

Measuring S to obtain C and S' to obtain Z gives:

$$\begin{aligned} \rho_{ZCP} = & \frac{1}{2}|0\rangle\langle 0|_Z \otimes |0\rangle\langle 0|_C \otimes \begin{pmatrix} c^2 & sc\sqrt{R} \\ sc\sqrt{R} & s^2R \end{pmatrix} \\ & + \frac{1}{2}|0\rangle\langle 0|_Z \otimes |1\rangle\langle 1|_C \otimes \begin{pmatrix} 0 & 0 \\ 0 & s^2(1-R) \end{pmatrix} \\ & + \frac{1}{2}|1\rangle\langle 1|_Z \otimes |0\rangle\langle 0|_C \otimes \begin{pmatrix} 0 & 0 \\ 0 & s^2(1-R) \end{pmatrix} \\ & + \frac{1}{2}|1\rangle\langle 1|_Z \otimes |1\rangle\langle 1|_C \otimes \begin{pmatrix} c^2 & -sc\sqrt{R} \\ -sc\sqrt{R} & s^2R \end{pmatrix}. \end{aligned} \quad (36)$$

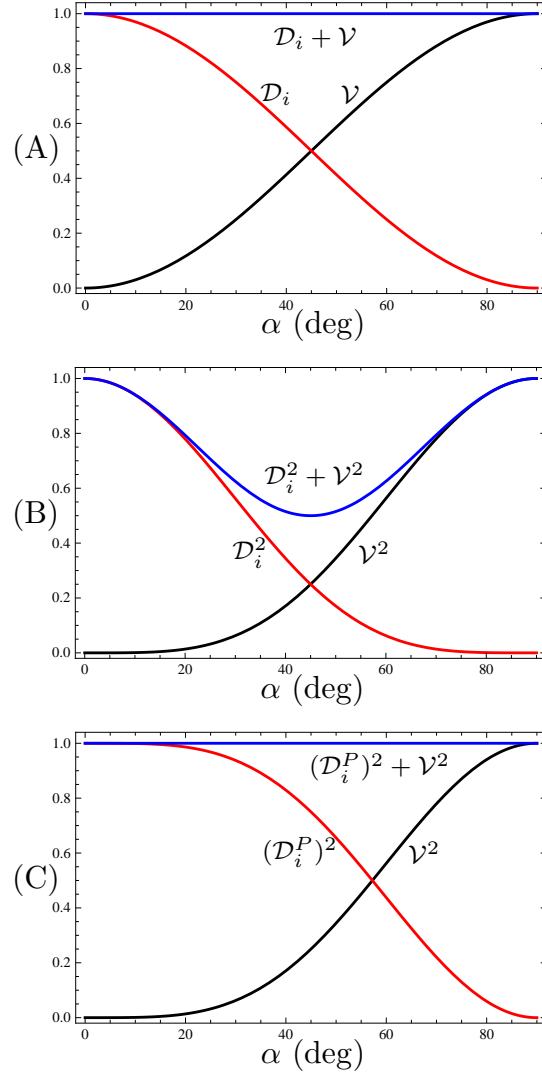


FIG. 5: For $R = 0.5$, we plot (A) \mathcal{V} , \mathcal{D}_i , and $\mathcal{V} + \mathcal{D}_i$; (B) \mathcal{V}^2 , \mathcal{D}_i^2 , and $\mathcal{V}^2 + \mathcal{D}_i^2$; and (C) \mathcal{V}^2 , $(\mathcal{D}_i^P)^2$, and $\mathcal{V}^2 + (\mathcal{D}_i^P)^2$. Notice that $\mathcal{V}^2 + (\mathcal{D}_i^P)^2 = 1$, showing that our novel WPDR is perfectly tight.

From ρ_{ZCP} we can compute the conditional states on CP associated with the different values of Z :

$$\sigma_{CP}^0 = \begin{pmatrix} c^2 & sc\sqrt{R} & 0 & 0 \\ sc\sqrt{R} & s^2R & 0 & s^2(1-R) \end{pmatrix},$$

$$\sigma_{CP}^1 = \begin{pmatrix} 0 & 0 & c^2 & -sc\sqrt{R} \\ 0 & s^2(1-R) & -sc\sqrt{R} & s^2R \end{pmatrix}.$$

Now we compute the distinguishability by using the fact that it is the trace distance between the conditional states. For the polarisation-enhanced distinguishability, this gives:

$$\mathcal{D}_i^P = \frac{1}{2} \|\sigma_{CP}^0 - \sigma_{CP}^1\|_1 = \sqrt{c^4 + s^4(2R-1)^2 + 2(sc)^2}.$$

Decohering σ_{CP}^0 and σ_{CP}^1 before calculating the trace norm leads to the decohered distinguishability:

$$\mathcal{D}_i^{P,dec} = c^2 + s^2 \cdot |2R - 1|.$$

Finally, to calculate the non-enhanced distinguishability we need to trace out the polarisation register:

$$\sigma_C^0 = \begin{pmatrix} c^2 + s^2 R & \\ & s^2(1 - R) \end{pmatrix} \quad \text{and} \quad \sigma_C^1 = \begin{pmatrix} s^2(1 - R) & \\ & c^2 + s^2 R \end{pmatrix}$$

and then

$$\mathcal{D}_i = \frac{1}{2} \|\sigma_C^0 - \sigma_C^1\|_1 = |c^2 + s^2(2R - 1)|.$$

Measuring \mathcal{D}_i^P

Measuring the polarisation-enhanced distinguishability \mathcal{D}_i^P requires polarization-resolving detectors on the output modes of the interferometer, as in Fig. 3. As defined, \mathcal{D}_i^P corresponds to measuring the optimal polarization basis on each output mode, i.e., optimally helpful for guessing which path the photon took. We now solve for this optimal polarization basis. We remark that varying the polarization measurement basis could be accomplished by varying the angle of a half-wave plate inserted just prior to the PBS's in Fig. 3.

Let us begin by expanding the guessing probability as follows

$$p_{\text{guess}}(Z_i|CP) = \Pr(C = 0)p_{\text{guess}}(Z_i|P, C = 0) + \Pr(C = 1)p_{\text{guess}}(Z_i|P, C = 1)$$

where $p_{\text{guess}}(Z_i|CP)$ is related to \mathcal{D}_i^P through Eq. (31). Since the probabilities $\Pr(C = 0)$ and $\Pr(C = 1)$ are experimentally accessible, we just need to find the measurements on P that would allow the experimenter to compute $p_{\text{guess}}(Z_i|P, C = 0)$ and $p_{\text{guess}}(Z_i|P, C = 1)$.

This is simply a two-state discrimination problem on a qubit, and the optimal solution is well-known [41]. Define the positive operators

$$\begin{aligned} \tau_P^{00} &= \text{Tr}_{ZC}((|0\rangle\langle 0|_Z \otimes |0\rangle\langle 0|_C \otimes \mathbb{1})\rho_{ZCP}) / \Pr(C = 0), \\ \tau_P^{10} &= \text{Tr}_{ZC}((|1\rangle\langle 1|_Z \otimes |0\rangle\langle 0|_C \otimes \mathbb{1})\rho_{ZCP}) / \Pr(C = 0), \\ \tau_P^{01} &= \text{Tr}_{ZC}((|0\rangle\langle 0|_Z \otimes |1\rangle\langle 1|_C \otimes \mathbb{1})\rho_{ZCP}) / \Pr(C = 1), \\ \tau_P^{11} &= \text{Tr}_{ZC}((|1\rangle\langle 1|_Z \otimes |1\rangle\langle 1|_C \otimes \mathbb{1})\rho_{ZCP}) / \Pr(C = 1). \end{aligned}$$

where τ_P^{00} and τ_P^{10} are the (unnormalised) conditional states on P associated $Z = 0$ and $Z = 1$ respectively, and both of which are conditioned on $C = 0$. Likewise τ_P^{01} and τ_P^{11} are conditioned on $C = 1$. From [41], the optimal polarisation bases to measure on the output modes 0 and 1 are, respectively, given by the eigenvectors of the following Hermitian operators:

$$\begin{aligned} \Lambda_P^0 &= \tau_P^{00} - \tau_P^{10}, \\ \Lambda_P^1 &= \tau_P^{01} - \tau_P^{11}. \end{aligned}$$

From the formula for ρ_{ZCP} in (36), we compute that these correspond to the following polarisation observables (represented as matrices in the $\{|H\rangle, |V\rangle\}$ basis):

$$\begin{aligned} O_P^0 &= \frac{1}{\mathcal{D}_i^P} \begin{pmatrix} 1 - 2s^2 R & 2sc\sqrt{R} \\ 2sc\sqrt{R} & -(1 - 2s^2 R) \end{pmatrix}, \\ O_P^1 &= \frac{1}{\mathcal{D}_i^P} \begin{pmatrix} 1 - 2s^2 R & -2sc\sqrt{R} \\ -2sc\sqrt{R} & -(1 - 2s^2 R) \end{pmatrix}. \end{aligned}$$

where we have normalised the observables such that they square to the identity. For example, choosing $R = 0.5$ and $\alpha = 45^\circ$ (corresponding to an equal superposition of BS₂ being "absent" and "present") gives

$$\begin{aligned} O_P^0 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \\ O_P^1 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}. \end{aligned}$$